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Semisimple rings of quotients in Morita contexts

Zhou Zhengping

Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

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Abstract

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Let $(R, {}_RU_S, {}_SV_R, S)$ be a Morita context with the trace ideals I in R and J in S, τ a Gabriel topology containing I on R-Mod, and τ' the corresponding Gabriel topology containing J on S-Mod. Necessary and sufficient conditions on τ and ${}_RU$ are given in order that the ring of quotients $\hat{S}$ of S with respect to τ' be semisimple artinian, simple artinian or a division ring respectively. Special cases include earlier works.

1. Introduction

It is well known that if two rings R and S are Morita equivalent, then many important ring-theoretical properties of one can be transferred to the other one, for example, being semisimple or simple (in this article, we always mean semisimple artinian or simple artinian respectively) [1, Corollaries 21.9 and 21.12]. There is also a lattice isomorphism between the Gabriel topologies on R-Mod and those on S-Mod, and the rings of quotients with respect to a pair of corresponding Gabriel topologies are also Morita equivalent [9, Proposition X.3.1]. That means that $\hat{R}$ is semisimple (simple) if and only if $\hat{S}$ is.

More generally, Leu and Hutchinson [6], and Kašu [4] proved that for any context $(R, {}_RU_S, {}_SV_R, S)$ with the trace ideals I in R and J in S, there is a lattice

Correspondence to: Zhou Zhengping, Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA. Email: zhou@math.uiowa.edu.

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isomorphism between the lattice $G(R)$ of all Gabriel topologies containing I on R-Mod, and the lattice $G(S)$ of all Gabriel topologies containing J on S-Mod. (From now on, whenever we talk about τ or τ', unless otherwise specified, we always mean that $\tau \in G(R)$, $\tau' \in G(S)$ being the corresponding one, denoted by $(\tau, \tau') \in (G(R), G(S))$, and that $\hat{}$ is the quotient functor on R-Mod or S-Mod with respect to τ and τ'.) Also in [6], under the assumption that the context is nondegenerate, sufficient conditions on R, ${}_RU$ and τ are given so that $\hat{S}$ be semisimple, simple and Morita equivalent to $\hat{R}$ [6, Theorem 4.1], which could be considered as a generalization of [2, Corollary 4.10] by Cunningham, Rutter and Turnidge. Müller also got a similar result [8, Proposition 21] where the nondegeneracy is even stronger than that above.

Also under the assumption that the context is nondegenerate, Hutchinson and Zelmanowitz gave necessary and sufficient conditions in terms of U and R for S to have a semisimple maximal left quotient ring [3, Theorem 5]. If, in addition, U is τ-free, Leu gave a necessary and sufficient condition so that $\hat{S}$ is a division ring [5, Theorem 11].

In this paper, we investigate an arbitrary Morita context, and give necessary and sufficient conditions in terms of U, R and τ so that $\hat{S}$ be semisimple, simple or a division ring respectively. It is also shown that Leu and Hutchinson's work [6, Theorem 4.1] and Leu's work [5, Theorem 11] are special cases of our result. Hutchinson and Zelmanowitz's work could be also considered as a special case of our work although their result is more precise due to the stronger assumption on the context. In doing so we employ the corresponding necessary and sufficient conditions on an arbitrary Gabriel topology of a ring R so that the ring of quotients $\hat{R}$ be semisimple, simple or a division ring from [10] and introduce some "bridges" to transfer these conditions from one ring to the other in a Morita context.

2. Preliminaries

We now collect some necessary facts about Gabriel topologies and torsion theories. For more details, the reader is referred to [9, Chapters VI and IX].

Throughout this paper, R and S are associative rings with identity, R-Mod and S-Mod denote respectively the categories of unital left R- and left S-modules. Modules, unless otherwise specified, are consistently left modules.

Definition 2.1. A *Morita context* $(R, {}_RU_S, {}_SV_R, S)$ consists of two rings R and S, two bimodules ${}_RU_S$ and ${}_SV_R$, and two bimodule homomorphisms $(-,-) : U \otimes_S V \to R$ and $[-,-] : V \otimes_R U \to S$ satisfying $u[v,u'] = (u,v)u'$ and $v(u,v') = [v,u]v'$ for all $u, u' \in U$ and $v, v' \in V$ with the images being I and J, respectively. Both I and J are ideals and are called the *trace* ideals of the context.

Definition 2.2. A non-empty set τ of left ideals of R is called a *Gabriel topology* on R-Mod if:

(T1) If $\mathcal{B}$ is a left ideal of R, $\mathcal{B} \supseteq \mathcal{A} \in \tau$, then $\mathcal{B} \in \tau$.

(T2) If $\mathcal{A}, \mathcal{B} \in \tau$, then $\mathcal{A} \cap \mathcal{B} \in \tau$.

(T3) If $\mathcal{A} \in \tau$, $r \in R$, then $(\mathcal{A} : r) \in \tau$.

(T4) If $\mathcal{B}$ is a left ideal of R, and there exists $\mathcal{A} \in \tau$ such that $(\mathcal{A} : b) \in \tau$ for every $b \in \mathcal{B}$, then $\mathcal{B} \in \tau$.

Definition 2.3. (1) A module M is called *τ-torsion* if $\mathrm{Ann}_R(m) \in \tau$ for every $m \in M$.

(2) A module M is called *τ-free* if $\mathrm{Ann}_M(\mathcal{A}) = 0$ for every $\mathcal{A} \in \tau$.

(3) A module M is called *τ-injective* if $\mathrm{Hom}(R, M) \to \mathrm{Hom}(\mathcal{A}, M) \to 0$ is exact under the canonical homomorphism for all $\mathcal{A} \in \tau$. M is *τ-closed* if it is both τ-free and τ-injective.

(4) A submodule K of M is called *τ-saturated* if M/K is τ-free. $\mathrm{Sat}_\tau(M)$ denotes the set of all τ-saturated submodules.

(5) A submodule K of M is called *τ-dense* if M/K is τ-torsion. $\tau(M)$ denotes the set of all τ-dense submodules. In particular, $\tau = \tau(R)$.

Proposition 2.4. (1) *Any module M has a largest τ-torsion submodule $T_\tau(M)$, and $T_\tau(M)$ is the smallest member of* $\mathrm{Sat}_\tau(M)$.

(2) *For any submodule K of M, let $\overline{K} = \{m \in M \mid Am \subseteq K \text{ for some } \mathcal{A} \in \tau\}$. This is the smallest τ-saturated submodule of M containing K, $K \in \tau(\overline{K})$ and $K = \overline{K}$ if and only if $K \in \mathrm{Sat}_\tau(M)$. Also $\overline{K_1 \cap K_2} = \overline{K_1} \cap \overline{K_2}$.*

(3) *For any module M, there is a τ-closed module $\hat{M}$ called the* module of quotients *of M with respect to τ. More specially, $\hat{R}$ is a ring called the* ring of quotients *of R with respect to τ. Moreover, there is a ring homomorphism ϕ from R to $\hat{R}$ such that $\phi(R) \in \tau(\hat{R})$, where $\hat{R}$ is regarded as a left R-module via ϕ. The full subcategory ${}_\tau L$ of all τ-closed modules is called the quotient category with respect to τ, and it is equivalent to a full subcategory of* $\hat{R}$-Mod.

(4) *For any module M, $\mathrm{Sat}_\tau(M)$ is a complete modular lattice with $K_1 \vee K_2 = \overline{K_1 + K_2}$, $K_1 \wedge K_2 = K_1 \cap K_2$, for all $K_1, K_2 \in \mathrm{Sat}_\tau(M)$.*

(5) *A τ-dense submodule K of a τ-free module M is essential in M.* □

3. Rings of quotients

Let τ be an arbitrary Gabriel topology on R-Mod. A set H of submodules of a module M will be called *cofinally finite* if given any $M' \in H$, there exists a finitely generated submodule $M'' \subseteq M'$ with $M'' \in \tau(M')$. We introduce the following results from [10] without proofs.

Proposition 3.1 (cf. [10, Theorem 2.1]). *Let τ be any Gabriel topology on R-Mod. Then the following are equivalent:*

(1) *$\hat{R}$ is semisimple.*

(2) $\mathrm{Sat}_\tau(R)$ *satisfies the a.c.c. (the ascending chain condition), and τ-closed modules are injective.* □

Proposition 3.2 (cf. [10, Proposition 2.3]). *Let τ be any Gabriel topology on R-Mod. Then τ-closed modules are all injective if and only if essential submodules of any τ-free module M are τ-dense in M.* □

The following are the "bridges" between these results and our results.

Theorem 3.3 (cf. [11, Theorems 2.3 and 2.4] and [12, Theorem 2.4]). *Let $(\tau,\tau') \in (G(R),G(S))$. Then:*

(1) *If M is injective and τ-free, then $\mathrm{Hom}_R(U,M)$ is injective and τ'-free.*

(2) *If N is injective and τ'-free, then $\mathrm{Hom}_S(V,N)$ is injective and τ-free.*

(3) *The functors $\mathrm{Hom}_R(U,-)$ and $\mathrm{Hom}_S(V,-)$ induce an equivalence between ${}_\tau L$ and ${}_{\tau'}L$.*

(4) *There is a lattice isomorphism between $\mathrm{Sat}_\tau(U)$ and $\mathrm{Sat}_{\tau'}(S)$.* □

Now we are able to show our main theorems.

Theorem 3.4. *Let $(\tau,\tau') \in (G(R),G(S))$. Then the following are equivalent:*

(1) *$\hat{S}$ is semisimple.*

(2) $\mathrm{Sat}_\tau(U)$ *satisfies the a.c.c., and τ-closed modules are injective modules.*

(3) $\mathrm{Sat}_\tau(U)$ *satisfies the a.c.c., and essential submodules of τ-free module M are τ-dense in M.*

(4) *$L(U)$ is τ-cofinally finite, and τ-closed modules are injective, where $L(U)$ denotes the lattice of submodules of U.*

Proof. (1) $\Leftrightarrow$ (2) By Proposition 3.1, $\hat{S}$ is semisimple if and only if $\mathrm{Sat}_{\tau'}(S)$ satisfies the a.c.c. and τ'-closed modules are injective, but by Theorem 3.3, this is equivalent to saying that $\mathrm{Sat}_\tau(U)$ satisfies the a.c.c. and τ-closed modules are injective. (2) $\Leftrightarrow$ (3) follows directly from Proposition 3.2. (3) $\Leftrightarrow$ (4) follows from [9, Proposition XIII.2.1]. □

In [6], under the following three assumptions: (1) the context $(R, {}_RU_S, {}_SV_R, S)$ is nondegenerate, i.e., U_S is faithful and $[V,u] \neq 0$ whenever $0 \neq u \in U$, (2) R and ${}_RU$ are both τ-free, (3) $d({}_RU)$, the Goldie dimension of ${}_RU$, is finite, it is shown that if $\hat{R}$ is simple, then $\hat{S}$ is, and they are Morita equivalent; if $\hat{R}$ is semisimple, then $\hat{S}$ is; if, in addition, (4) ${}_RU$ is faithful, then they are Morita equivalent [6, Theorem 4.1]. However, we can show, as a consequence of Theorem 3.4, the same result under only the assumption that $d(U/T_\tau(U))$ is finite (Theorem 3.8).

First of all we need the following lemmas.

Lemma 3.5. *For any* τ *on* R-Mod, *the following are equivalent*:

(1) $\hat{R}\mathcal{A} = \hat{R}$ *for all* $\mathcal{A} \in \tau$.

(2) *For any left ideal* $\mathcal{B}$ *of* $\hat{R}$, *if* $\mathcal{B} \in \tau(\hat{R})$, *then* $\mathcal{B} = \hat{R}$.

Proof. (1) $\Rightarrow$ (2) Since $\mathcal{B} \in \tau(\hat{R})$, we have $\hat{R}/\mathcal{B}$ is τ-torsion. Therefore $R/\phi^{-1}(\mathcal{B})$ is τ-torsion since it is isomorphic to an R-submodule of $\hat{R}/\mathcal{B}$. This means that $\phi^{-1}(\mathcal{B}) \in \tau$. Then we have $\hat{R} = \hat{R}\phi^{-1}(\mathcal{B}) = \hat{R}\phi(\phi^{-1}(\mathcal{B})) \subseteq \mathcal{B}$ since $\mathcal{B}$ is a left ideal of $\hat{R}$ and $\phi(\phi^{-1}(\mathcal{B})) \subseteq \mathcal{B}$. Thus $\hat{R} = \mathcal{B}$.

(2) $\Rightarrow$ (1) $\phi(R)/\phi(\mathcal{A}) \cong R/\phi^{-1}(\phi(\mathcal{A})) = R/(\mathcal{A} + T_\tau(R))$ since $T_\tau(R)$ is the kernel of ϕ. We have $\phi(\mathcal{A}) \in \tau(\phi(R))$. But $\phi(R) \in \tau(\hat{R})$; thus $\phi(\mathcal{A}) \in \tau(\hat{R})$. However, $\hat{R}\mathcal{A} = \hat{R}\phi(\mathcal{A})\ \phi(\mathcal{A})$ and so $\hat{R}\mathcal{A} \in \tau(\hat{R})$. By hypothesis, $\hat{R}\mathcal{A} = \hat{R}$. □

Recall that a Gabriel topology τ on R-Mod is said to be perfect if and only if the two categories ${}_\tau L$ and $\hat{R}$-Mod are equivalent; and this is true if and only if $\hat{R}\mathcal{A} = \hat{R}$ for all $\mathcal{A} \in \tau$ [9, Proposition XI.3.4(g)].

Lemma 3.6. *For any* τ *on* R-Mod, *if* $\hat{R}$ *is semisimple, then* τ *is perfect.*

Proof. Let $\mathcal{B}$ be any left ideal of $\hat{R}$, and $\mathcal{B} \in \tau(\hat{R})$. Then $\hat{R}/\mathcal{B}$ is τ-torsion. Since $\hat{R}$ is τ-free, we have $\mathcal{B}$ is essential in $\hat{R}$. However $\hat{R}$ has no proper essential left ideal since it is semisimple. This implies $\mathcal{B} = \hat{R}$. Now we apply Lemma 3.5. □

Lemma 3.7. *For any Gabriel topology* τ *on* R-Mod, *and any* τ-*free* R-*module* M, $d(M) = d(\hat{M}) = d({}_{\hat{R}}\hat{M})$. *Therefore, for any* R-*module* M, $d(\hat{M}) = d({}_{\hat{R}}\hat{M}) = d(M/T_\tau(M))$.

Proof. Let $C(M)$ denote the set of all complement submodules, i.e. essential closed submodules, of M, and $C(\hat{M})$ that of $\hat{M}$. First we claim that if M is τ-free, then $C(M) \subseteq \mathrm{Sat}_\tau(M)$. In fact, if $M' \in C(M)$, then $\overline{M'}/M'$ is τ-torsion. But M, and hence also $\overline{M'}$, is τ-free. Hence M' is essential in $\overline{M'}$, and so $\overline{M'} = M'$ and $M' \in \mathrm{Sat}_\tau(M)$ as required. Secondly let $M_1 \subset M_2 \subset \cdots \subset M_n$ be a chain of elements in $\mathrm{Sat}_\tau(M)$. Let M'_n be a relative complement for M_n in M. Since $M'_n \cap M_{n-1} \subseteq M'_n \cap M_n = 0$, we can enlarge M_n to a relative complement M'_{n-1} for M_{n-1}. Continuing in this manner, we obtain a chain of elements in $C(M)$, $M'_n \subseteq M'_{n-1} \subseteq \cdots \subseteq M'_1$. If $M'_i = M'_{i-1}$, then we have $M_i + M'_i = M_{i-1} + M'_{i-1}$, $M_i \cap M'_i = M_{i-1} \cap M'_{i-1} = 0$, $M_{i-1} \subseteq M_i$. Therefore, $M_{i-1} = M_i$. This implies that the maximal length of a chain of elements in

$\mathrm{Sat}_\tau(M)$ is the same as the maximal length of a chain of elements in $C(M)$, which is exactly $d(M)$ by [7, p. 52, Corollary (v)]. on the other hand, $\mathrm{Sat}_\tau(M)$ is lattice isomorphic to $\mathrm{Sat}_\tau(\hat{M})$, therefore $d(M) = d(\hat{M})$ since $\hat{M}$ is τ-free, also. Now if we let $\bar{\tau}$ be the Gabriel topology on $\hat{R}$-Mod corresponding to τ, then ${}_{\hat{R}}\hat{M}$ is $\bar{\tau}$-free, and therefore by the above argument and result $d({}_{\hat{R}}\hat{M})$ is the maximal length of a chain of elements in $\mathrm{Sat}_{\bar{\tau}}({}_{\hat{R}}\hat{M})$, which is the same as $\mathrm{Sat}_\tau(\hat{M})$ (cf. Proposition 4.2. on p. 207, and the second paragraph on p. 218 in [9]). So we have $d(M) = d(\hat{M}) = d({}_{\hat{R}}\hat{M})$ if M is τ-free. Now for any M, $M/T_\tau(M)$ is τ-free, and $M/\widehat{T_\tau(M)} = \hat{M}$, the last statement follows. □

Now we have the following theorem:

Theorem 3.8. *Suppose that* $(\tau, \tau') \in (G(R), G(S))$, $d(U/T_\tau(U)) < \infty$, *and* $\hat{R}$ *is semisimple. Then* $\hat{R}$ *and* $\hat{S}$ *are Morita equivalent. Hence* $\hat{S}$ *is semisimple, and is simple if and only if* $\hat{R}$ *is simple.*

Moreover, $\mathrm{Hom}_{\hat{R}}(\hat{U}, -)$ *and* $\mathrm{Hom}_{\hat{S}}(\hat{V}, -)$ *is a pair of functors giving an equivalence of* $\hat{R}$-Mod *and* $\hat{S}$-Mod.

Proof. Notice that if $\hat{S}$ is semisimple, then it follows easily that $\hat{R}$ and $\hat{S}$ are Morita equivalent via $\mathrm{Hom}_R(U,-)$ and $\mathrm{Hom}_S(V,-)$ from Lemma 3.6, Theorem 3.3(3), and [9, Proposition XI.3.4(a)], and hence $\hat{R}$ is simple if and only if $\hat{S}$ is simple. However, by [9, Corollary IX.1.10, Proposition IX.1.11] $\mathrm{Hom}_R(U,-) \cong \mathrm{Hom}_{\hat{R}}(\hat{U},-)$ and $\mathrm{Hom}_S(V,-) \cong \mathrm{Hom}_{\hat{S}}(\hat{V},-)$ on ${}_\tau L$ and ${}_{\tau'}L$ respectively. So it suffices to show that $\hat{S}$ is semisimple. Since $\hat{R}$ is semisimple, then every τ-closed module is injective by Proposition 3.1. We show next that $\mathrm{Sat}_\tau(U)$ satisfies the a.c.c. $\mathrm{Sat}_\tau({}_R U)$ is lattice isomorphic to $\mathrm{Sat}_\tau({}_R\hat{U})$ by [9, Proposition IX.4.3]. But $\mathrm{Sat}_\tau({}_R\hat{U})$ consists of τ-closed submodules of ${}_R\hat{U}$ by [9, Proposition IX. 4.2]. This is equivalent to a sublattice of all $\hat{R}$-submodules of ${}_{\hat{R}}\hat{U}$ by Proposition 2.4(3). However, $d({}_{\hat{R}}\hat{U}) = d(U/T_\tau(U) < \infty$ by Lemma 3.7, and $\hat{R}$ is semisimple; so we know that ${}_{\hat{R}}\hat{U}$ satisfies the a.c.c. on submodules. It follows that $\mathrm{Sat}_\tau({}_R U)$ satisfies the a.c.c. Then $\hat{S}$ is semisimple by Theorem 3.4. □

Corollary 3.9. *Suppose that* $(\tau, \tau') \in (G(R), G(S))$, *and* $d(U/T_\tau(U))$ *and* $d(V/T_\tau(V))$ *are both finite. Then* $\hat{R}$ *is semisimple, or simple, if and only if* $\hat{S}$ *is semisimple, or simple, respectively.*

Proof. By the symmetry of a Morita context and Theorem 3.8. □

A related result is [8, Proposition 21].

Theorem 3.10. *Let* $(\tau, \tau') \in (G(R), G(S))$, *and* $\hat{S}$ *be semisimple. Then the following are equivalent*:

(1) $\hat{S}$ *is simple.*

(2) $T_{\tau_1}(U) = T_\tau(U)$ *for any* $\tau_1 \supseteq \tau$.

(3) $\hat{U} = {}_{\tau_1}\hat{U}$ *for any* $\tau_1 \supseteq \tau$, *where* ${}_{\tau_1}\hat{U}$ *is the module of quotients of* U *with respect to* τ_1.

Proof. (1) $\Leftrightarrow$ (2) By [10, Theorem 5.4], $\hat{S}$ is simple if and only if $T_{\tau'}(S)$ is a maximal torsion ideal, i.e., if τ'_1 is any Gabriel topology on S-Mod (not necessarily in $G(S)$), and $T_{\tau'_1}(S) \supseteq T_{\tau'}(S)$, then $T_{\tau'_1}(S) = T_{\tau'}(S)$. However, from the proof it is easy to see that it is also equivalent to that $T_{\tau'}(S) = T_{\tau'_1}(S)$ for any $\tau'_1 \supseteq \tau'$ (hence $\tau'_1 \in G(S)$ if $\tau' \in G(S)$. From [12, Theorem 2.4], there are lattice isomorphisms F and F^{-1} between $\mathrm{Sat}_\tau(U)$ and $\mathrm{Sat}_{\tau'}(S)$ for any $(\tau, \tau') \in (G(R), G(S))$, and $F(T_\tau(U)) = \{s \in S \mid Us \subseteq T_\tau(U)\} = T_{\tau'}(S)$, $F^{-1}(T_\tau(S)) = \{u \in U \mid [V, u] \subseteq T_{\tau'}(S)\} = T_\tau(U)$. Now if $\tau_1 \supseteq \tau$, and $T_{\tau'_1}(S) = T_{\tau'}(S)$, then $T_\tau(U) = F^{-1}(T_{\tau'}(S)) = F^{-1}(T_{\tau'_1}(S)) = T_{\tau_1}(U)$. Conversely if $\tau'_1 \supseteq \tau'$, and $T_\tau(U) = T_{\tau_1}(U)$, then $T_{\tau'}(S) = F(T_\tau(U)) = F(T_{\tau_1}(U)) = T_{\tau'_1}(S)$. This shows that $T_{\tau'}(S) = T_{\tau'_1}(S)$ for any $\tau'_1 \supseteq \tau'$ if and only if $T_\tau(U) = T_{\tau_1}(U)$ for any $\tau_1 \supseteq \tau$.

(2) $\Rightarrow$ (3) If $T_{\tau_1}(U) = T_\tau(U), \tau_1 \supseteq \tau$, then $\hat{U}$ and ${}_{\tau_1}\hat{U}$ are τ- and τ_1-injective hulls of $U/T_\tau(U)$ respectively, which are both contained in $E(U/T_\tau(U)$, the injective hull of $U/T_\tau(U)$. Since $\tau_1 \supseteq \tau$, we have $U/T_\tau(U) \subseteq \hat{U} \subseteq {}_{\tau_1}\hat{U} \subseteq E(U/T_\tau(U))$. But $\hat{U}$ is injective by Theorem 3.4, and hence $\hat{U} = {}_{\tau_1}\hat{U}$

(3) $\Rightarrow$ (2) If $\tau_1 \supseteq \tau$, then the τ_1-torsion module $T_{\tau_1}(U)/T_\tau(U)$ can be considered as a submodule of $\hat{U} = {}_{\tau_1}\hat{U}$, which is also τ_1-free. So $T_{\tau_1}(U)/T_\tau(U) = 0$ and hence $T_{\tau_1}(U) = T_\tau(U)$. $\square$

Combining Theorems 3.8 and 3.10, we obtain the following corollary:

Corollary 3.11. *Let* $(\tau, \tau') \in (G(R), G(S))$, $d(U/T_\tau(U)) < \infty$, *and* $\hat{R}$ *be semisimple. Then the following are equivalent*:

(1) $\hat{R}$ *is simple.*

(2) $\hat{S}$ *is simple.*

(3) $T_{\tau_1}(U) = T_\tau(U)$ *for any* $\tau_1 \supseteq \tau$.

(4) $\hat{U} = {}_{\tau_1}\hat{U}$ *for any* $\tau_1 \supseteq \tau$, *where* ${}_{\tau_1}\hat{U}$ *is the module of quotients of* U *with respect to* τ_1. $\square$

Next we give a necessary and sufficient condition on τ and U so that $\hat{S}$ is a division ring. Let τ be a Gabriel topology on R-Mod, and $M \in R$-Mod. Then M is said to be τ-simple if $\mathrm{Sat}_\tau(M) = \{T_\tau(M), M\}$. Therefore, we have the following:

Theorem 3.12. *Let* $(\tau, \tau') \in (G(R), G(S))$. *Then* $\hat{S}$ *is a division ring if and only if* U *is* τ*-simple.*

Proof. Combine our Theorem 3.3(4) and [10, Theorem 5.5]. $\square$

This is a significant improvement of [5, Theorem 11], in which it is shown that if the context is derived and nondegenerate, U is τ-free, then $\hat{S}$ is a division ring if and only if U is a support of τ on R-Mod, i.e., for any $0 \neq U' \subseteq U$, $U' \in \tau(U)$, which is equivalent to that U is τ-simple and τ-free.

In [3, Theorem 5], Hutchinson and Zelmanowitz showed the equivalence of (1) and (2) of the next theorem.

Theorem 3.13. *Suppose the context* $(R, {}_RU_S, {}_SV_R, S)$ *with the trace ideals* I *and* J *is nondegenerate,* τ' *is the dense topology on* S-Mod, τ *is the corresponding Gabriel topology in* $G(R)$. *Then the following are equivalent*:

(1) S *has a semisimple maximal left quotient ring.*

(2) $Z(U) = 0$, *and* $d(U) < \infty$, *where* $Z(U)$ *is the singular submodule of* U.

(3) $\mathrm{Sat}_\tau(U)$ *satisfies the a.c.c., and* τ*-closed modules are injective.*

(4) $\mathrm{Sat}_\tau(U)$ *satisfies the a.c.c., and essential submodules of* τ*-free module* M *are* τ*-dense in* M.

(5) $L(U)$ *is* τ*-cofinally finite, and* τ*-closed modules are injective, where* $L(U)$ *denotes the lattice of submodules of* U.

Proof. Since the context $(R, {}_RU_S, {}_SV_R, S)$ is nondegenerate, then ${}_SS$ is τ_J-free, where τ_J is the smallest one in $G(S)$. In fact, τ_J is the Gabriel topology determined by the trace ideal J, and an S module N is τ_J-free if and only if $Jn \neq 0$ whenever $0 \neq n \in N$. If $s \in {}_SS$ and $Js = 0$, then $[V, U]s = 0$ and so $[V, Us] = 0$. Therefore, $Us = 0$ and $s = 0$ by the nondegeneracy. Let τ' be the dense topology on S-Mod, which is the largest Gabriel topology on S-Mod such that S is torsion free, so $\tau' \in G(S)$. Suppose that $\hat{S}$ is the rings of quotients of S with respect to τ', which is the maximal left quotient ring of S, τ is the corresponding Gabriel topology on R-Mod, then we apply Theorem 3.4. $\square$

Finally, we give an example to which our results can apply and the old cannot. Let R be the ring of matrices of the form

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$$

where $x, y \in \mathbb{C}$, the complex field, and $z \in \mathbb{R}$, the real number field. Then

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is a basic set of primitive idempotents for R, the Jacobson radical $J(R)$ is

$$\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}$$

and $R_1 = Re_1/J(R)e_1, R_2 = Re_2/J(R)e_2 = Re_2$ is an irredundant set of representatives of the simple left R-modules. Let $U = R_1 \oplus R_2$. Consider the derived context from ${}_RU$: $(R, {}_RU_S, {}_SV_R = \mathrm{Hom}_R(U,R),\ S = \mathrm{End}_R(U))$, then it is easy to see that the trace ideal I of U in R is

$$\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R},$$

$V = \mathrm{Hom}_R(U,R) = \mathrm{Hom}_R(U,I) \cong \mathrm{Hom}_R(R_2,I) \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, and $S = \mathrm{Hom}_R(U,U) \cong \mathbb{C} \times \mathbb{R}$. We now consider the Gabriel topology τ_I determined by I and the corresponding one τ_J determined by J, the trace ideal in S. Since $I^2 = I$, for any module M, it is τ_I-torsion if and only if $IM = 0$, and $\hat{M} = \mathrm{Hom}_R(I, \mathrm{Hom}_R(I,M))$ (cf. [9, p. 200]). Also notice that $IR_1 = 0$, then it is easy to see that $T_{\tau_I}(U) = R_1 \neq 0$, $d(U/T_{\tau_I}(U)) < \infty$, $d(V/T_{\tau_J}(V)) < \infty$ and $\hat{R} \cong M_3(\mathbb{R})$. Therefore we can apply Theorem 3.8 or Corollary 3.9 to conclude that $\hat{R}$ and $\hat{S}$ are Morita equivalent, hence $\hat{S}$ is simple, and Theorem 3.10 or Corollary 3.11 to conclude that for any $\tau \in G(R)$, $T_\tau(U) = T_{\tau_I}(U)$ and ${}_\tau\hat{U} = {}_{\tau_I}\hat{U}$. Since U is clearly τ_I-simple, we can also directly apply Theorem 3.12 to know that $\hat{S}$ is a division ring. In fact, $J = \mathbb{R}$ and $\hat{S} \cong \mathbb{R}$. But the old results ([5, Theorem 11] and [6, Theorem 4.1]) cannot apply to this context since U is not τ_I-free. Moreover, the context is degenerate because from the definition of $[V,U]$ and the fact that $R_1 \ncong R_2$, $[V,u] = 0$ for any $u \in R_1$. U is not faithful either since $\mathrm{Ann}_R(U) = J(R) \neq 0$. However, $T_{\tau_I}(R) = 0$. We also point out that $\hat{R}$ is the maximal left quotient ring of R while $\hat{S}$ is not the maximal left quotient ring of S.

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References

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules (Springer, New York, 1974).

[2] R.S. Cunningham, E.A. Rutter and D.R. Turnidge, Rings of quotient of endomorphism rings of projective modules, Pacific J. Math. 41 (1972) 647–668.

[3] J. Hutchinson and J.M. Zelmanowitz, Quotient rings of endomorphism rings of modules with zero singular submodule, Proc. Amer. Math. Soc. 35 (1972) 16–20.

[4] A.I. Kašu, Morita contexts and torsions of modules, Math. Zametki 28 (4) (1980) 491–499.

[5] H. M. Leu, The ring of quotients of a module endomorphism ring, Tamkang J. Math. 7 (1976) 77–86.
[6] H.M. Leu and J.J. Hutchinson, Kernel functors and quotient rings, Bull. Inst. Math. Acad. Sinica 5 (1) (1977) 145–155.
[7] J.C. McConnell and J.C. Robson, Noncommutative Noetherian Rings (John Wiley and Sons, Chichester, UK, 1987).
[8] B.J. Müller, The quotient category of a Morita context, J. Algebra 28 (1974) 389–407.
[9] B. Stenström, Rings of Quotients (Springer, Berlin, 1975).
[10] J.M. Zelmanowitz, Semisimple rings of quotients, Bull. Austral. Math. Soc. 19 (1978) 97–115.
[11] Z. Zhou, Equivalence and duality of quotient categories, J. Algebra 143 (1991) 144–155.
[12] Z. Zhou, Relative correspondence theorem for a Morita context, Submitted for publication.